Quantum Formalism with State-Collapse and Superluminal Communication

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Abstract

Given the collapse hypothesis (CH) of quantum measurement, EPR-type correlations along with the hypothesis of the impossibility of superluminal communication (ISC) have the effect of globalizing gross features of the quantum formalism making them universally true. In particular, these hypotheses imply that state transformations of density matrices must be linear and that evolution which preserves purity of states must also be linear. A gedanken experiment shows that lorentz covariance along with the second law of thermodynamics imply a non-entropic version of ISC. Partial results using quantum logic suggest, given ISC and a version of CH, a connection between lorentz covariance and the covering law. These results show that standard quantum mechanics is structurally unstable, and suggest that viable relativistic alternatives must question CH. One may also speculate that some features of the hilbert-space model of quantum mechanics have their origin in space-time structure.

1 Introduction

Inspired by the paper of Einstein, Podolsky, and Rosen [1], and more recently by the controversy surrounding the Bell inequalities [2, 3], many researchers have tried to interpret long-range quantum correlations as resulting

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from some sort of action-at-a-distance. A natural question then arises as to whether such correlations can be used for superluminal communication. Several authors [4, 5, 6, 7] argued that no such communication is possible since the statistical behavior of any detector placed on one arm of an EPR apparatus is completely independent of what is done on the other arm. Nick Herbert [8] argued that if one can construct a "photon duplicator" which reproduces exactly an incoming photon state, then superluminal communication is possible. This argument quickly provoked several rebuttals [9, 10, 11] to the effect that no linear state transformer can clone an arbitrary photon state. It seems therefore that ordinary hilbert-space quantum mechanics precludes the use of EPR-type correlations for superluminal communication. We address in this section the reciprocal question: under the hypothesis of the impossibility of superluminal communication, what can one deduce about the behavior of detectors and state transformers? It is a common view that superluminal communication should not be possible as it raises serious questions of relativistic covariance and causality. These will be further considered in Section 4.

To be able to attack the question at hand we shall initially work with the hypothesis that there are possibly some physical processes that may not conform to the usual quantum mechanical description but that these are specific to very particular situations, whereas for the vast majority of other processes (including all those experimentally studied up to now), any deviation from normal quantum mechanical predictions is below present experimental precision. Thus one may posit that there may be a state transformer, such as the hypothetical photon cloner mentioned above, that acts in a non-linear fashion, and that such a transformer may take part in an experimental arrangement in which normal quantum mechanical description is adequate for processes not involving it. Explicitly the hypothesis is then that, in any given inertial frame, up to the use of an unconventional device, the usual quantum mechanical reasoning can be used, including the projection rule. Up to such a moment, ordinary quantum mechanics determines what the physical state is. At the point of using the unconventional device we of course must posit what would happen (a photon would be cloned in the above cited example). We shall call allowing the situations described above the *neighborhood hypothesis* (NH) since what we are describing, informally speaking, is a situation that would neighbor standard quantum mechanics in the set of all possible physical theories.

We shall show in sections 2 and 3 that under NH, certain types of devi-

ation, specifically non-linearities and lack of true randomness of outcomes, allow for superluminal signals. These theories are thus to be ruled out if we assume the impossibility of superluminal communication (ISC). This makes ordinary quantum mechanics a structurally unstable theory. This is important as many proponents of modifications to ordinary quantum mechanics are in fact implicitly assuming NH and so face a real risk of coming into conflict with special relativity.

It must be emphasized that NH is essentially an assumption about formalism and not about interpretation. It is an alteration of the formalism that is generally part of what is known as the Copenhagen interpretation, but we make no interpretational hypotheses. Though state collapse is used, we make no assumption as to its ontological nature, only that it is a legitimate calculating device for joint probabilities of events. In the end what we are saying is that joint probabilities cannot be calculated by certain rules if ISC is to be maintained. This makes our results basically interpretation independent.

It should also be noted that part of our understanding about the standard formalism is that it is capable of giving account of a relativistically covariant theory. This is not straightforwardly obvious given the instantaneous nature of wave function collapse [12, 13], but this does not preclude lorentz covariance of observable quantities. What the standard formalism lacks is thus *manifest* covariance while being able to provide for covariance of measurable magnitudes. It is precisely this fact that makes the theory structurally unstable, for a perturbation in the formalism is likely to make the manifest non-covariance capable of producing real effects, such as superluminal communication.

In section 4 we relate ISC and the second law of thermodynamics showing that under certain hypotheses superluminal communication can be used to foil the second law. In section 5 we argue that ISC, along with the projection postulate, can be used as supporting evidence for assuming certain axioms in the foundations of quantum mechanics thus suggesting that quantum mechanics owes some of its aspects to space-time structure. In the last section we draw some brief conclusions from these considerations.

2 The projection postulate and superluminal communication.

We shall assume the existence of an EPR-type apparatus by which a physical system decomposes into two parts which then separate in such a way that future measurements on each part separately can be performed at space-like separation. To avoid the complications of Bose or Fermi symmetrization, we shall assume the two parts are not identical, nevertheless we posit that the internal degrees of freedom of each part are described by finite-dimensional hilbert spaces of the same dimension which is at least three. In what follows we assume that the wave function factors into a product of the spatial part and the internal part. We focus only on the internal factor. Let $e_1, \ldots, e_N$ and $f_1, \ldots, f_N$ be orthonormal bases for the hilbert spaces of the internal degrees of freedom. We shall then assume that one can prepare the composite system in the state $\Psi = (1/\sqrt{N})(f_1 \otimes e_1 + f_2 \otimes e_2 + \cdots + f_N \otimes e_N)$. We also assume that for one arm of the apparatus (hereafter referred to as arm A), given any basis $h_1, \ldots, h_N$ for the corresponding hilbert space, we can measure a non-degenerate observable with this eigenbasis. Swift and Wright [14] have argued that for certain spin systems one can physically prepare a state corresponding to any ray in the corresponding hilbert space, and construct an actual apparatus corresponding to any self-adjoint operator. We shall thus generally assume that there are apparatus corresponding to any self-adjoint operator that we consider. We chose a reference frame in which the observation on arm A is temporally prior to that on arm B and, under the Neighborhood Hypothesis, and the assumption that superluminal communication is impossible, determine what must hold at arm B.

We first note that $e_i = \sum u_{ij} h_j$ for some unitary matrix u. This means that $\Psi = (1/\sqrt{N}) \sum g_j \otimes h_j$ where $g_j = \sum u_{ij} f_i$ defines another orthonormal basis, and any basis can be so constructed. Under the usual projection postulate, if we now perform observations corresponding to a non-degenerate self-adjoint operator with eigenbasis $h_1, \ldots, h_N$ on arm A, then subsequently the state collapses to a mixture in equal proportions of product states $g_j \otimes h_j$, and consequently the part of the system that is in arm B can be construed as being, in equal proportions, in the well defined quantum states g_j. Let us now locate a detector on arm B of the apparatus and suppose that the detection rate for a pure state ϕ is $D(\phi)$. The detection rate on arm B is then $(1/N) \sum D(g_j)$. For superluminal communication to be impossible

this number must be independent of the orthonormal basis $g_1, \ldots, g_N$ since otherwise the person stationed on arm A could, by changing his observable, induce a change in the detection rate on arm B at a space-like separation. This means by Gleason's theorem [15], that there is a positive operator R such that $D(\phi) = (\phi, R\phi)$. Since an observable can be construed as the simultaneous action of a number of mutually exclusive compatible detectors (one for each "pointer position"), we also conclude that on arm B, the mean value of any observable in state ϕ must be given by the usual quantum mechanical formula $(\phi, A\phi)$ for some self-adjoint operator A. As far as the expectation values of observables are concerned, arm B must then also follow the usual rules. Let us now consider on arm B a state transformer that transforms a state ϕ into another, possibly mixed state. Let us indicate this action on the corresponding density matrices as $\rho_\phi \mapsto T\rho_\phi$, where $\rho_\phi = (\phi, \cdot)\phi$ and T is *a-priori* an arbitrary map. If, after we subject the incoming state to transformation T, we then perform an observation corresponding to a conventional quantum observable represented by the operator A, the resultant expected value is $Tr(AT\rho_\phi)$ and by our previous result this must be of the form $(\phi, \tau(A)\phi)$ for some self-adjoint operator $\tau(A)$. Choose now an orthonormal basis $k_1, \ldots, k_P$ for the space in which the transformed states lie (which may be different from the space in which ϕ lies, as would be the case for the putative "photon cloner"). Setting $A = (1/2)((k_p, \cdot)k_q + (k_q, \cdot)k_p)$ and then $A = (1/2i)((k_p, \cdot)k_q - (k_q, \cdot)k_p)$ one deduces that $\mathrm{Re}\,(k_p, T\rho_\phi k_q) = (\phi, M_{pq}\phi)$ and $\mathrm{Im}\,(k_p, T\rho_\phi k_q) = (\phi, N_{pq}\phi)$ for some self-adjoint operators M_{pq} and N_{pq} with $M_{pq} = M_{qp}$ and $N_{pq} = -N_{qp}$. If we now set $L_{pq} = M_{pq} + iN_{pq}$ we have $(k_p, T\rho_\phi k_q) = (\phi, L_{pq}\phi)$ and finally

$$T\rho_\phi = \sum(\phi, L_{pq}\phi)(k_q, \cdot)k_p = \sum Tr(L_{pq}\rho_\phi)(k_q, \cdot)k_p$$

but this is a *linear* function of ρ_ϕ. We thus conclude that at arm B any density matrix state transformer must be linear. Finally, suppose the state transformer does not turn pure states into mixed states, then we can write the transformation as $\phi \mapsto S\phi$. Now by the previous result one has that $(\phi, \cdot)\phi \mapsto (S\phi, \cdot)S\phi$ must be a restriction of a linear map and given by the formula above. We have been able to prove a theorem (see the appendix) which states that in this case, if the range of S is not confined to one ray, $(S\phi, \cdot)S\phi = (C\phi, \cdot)C\phi$ where C is either a linear or an anti-linear t ransformation. Non-degenerate state transformers on arm B that do not create mixed states from pure ones must therefore be representable by either linear or anti-linear mappings. Summing up:

Under the Neighborhood Hypothesis and normal quantum mechanical processes in relation to one arm of an EPR-type apparatus, and assuming the impossibility of superluminal communication, then at the distant other arm 1) the mean value of observables must be given by the usual quantum mechanical expectation value formula 2) density-matrix state transformers must be linear and 3) vector state transformers whose range contains more than one ray must be either linear or anti-linear.

The wayward transformers are those of the form $S\phi = \theta(\phi)(\phi, D\phi)^{1/2}\psi$ for some positive operator D, vector ψ, and unimodular function θ. Note that $(S\phi, AS\phi)$ is just $(\phi, D\phi)(\psi, A\psi)$ and so no measurement on $S\phi$ provides any more information about ϕ than that already given by the matrix elements $(\phi, D\phi)$. Hence the transformer is no more than the observable D in disguise, no real information about ϕ being passed on to the transformed state ψ. This degenerate case is thus of little further interest.

The analysis of this section doesn't go through for quantum systems described by a two-dimensional Hilbert space as Gleason's theorem does not then hold. Nevertheless, as Nick Herbert's [8] example shows, certain non-linearities can still lead to superluminal communication. We have not determined what the exact restrictions on non-linearities are in the two-dimensional case as these are specialized physical situations. Although logically tenable, the idea that physics changes radically once a physical system described by a two dimensional Hilbert space gets isolated from the environment, is to allow what must be deemed a rather bizarre situation. From the still informal viewpoint of a neighborhood of standard quantum theory, such theories would not constitute a generic situation and are thus left out of this first analysis.

What this all suggests is that some of the gross features of the quantum mechanical formalism are intimately connected to spatio-temporal relations. In particular, if part of the world follows standard rules, then the rest must follow suit if superluminal communication is to be ruled out. There cannot be any "small deviations". To what extent the impossibility of superluminal communication, *per se*, limits any possible physical theory is not clear. One is aware of the great difficulties in trying to justify the quantum formalism on clear *a-priori* physical grounds. The existing axiomatic schemes, of which the two most developed are due to Ludwig [16] and Piron [17], all suffer from the defect that the crucial axioms are by no means compelling nor even clear

as to their true physical content. There is now the intriguing possibility that some spatio-temporal hypothesis such as the impossibility of superluminal communication could result in hilbert-space quantum mechanics in a more natural way. We begin to address this question in Section 5 of this paper.

Various researchers have expressed the hope that some paradoxes of quantum mechanics (especially those connected with "Schrödinger's Cat", that is, measurement theory) would eventually be resolved by non-linearities (which presumably become more important as the number of particles increases) in the time evolution [18]. We see now that any such non-linearity carries with it the real possibility of superluminal communication and its concomitant problems. Steven Weinberg [19, 20] has proposed a non-linear quantum theory which in principle could be tested experimentally by effects such as spectral line broadening. Any such effect could be immediately used to build a superluminal communication device. That this is possible for the Weinberg theory has already been shown by Gisin [21] who has independently come to conclusions similar to the ones presented in this paper. That there is a connection between quantum evolution and superluminal communication has been pointed out in various articles by Gisin [22, 23, 24, 21] and by Pearle [25, 26]. Our result shows the generality of such a connection, it essentially is a consequence of the projection postulate and the structure of hilbert space, concretely Gleason's theorem.

Deeming superluminal communication undesirable, one can try to avoid it, while maintaining non-linearity, by modifying the projection postulate. While logically plausible, there are at present no known examples of a relativistic non-linear quantum mechanics. We discuss elsewhere [27] the possibility of such a theory. In recent years there has been a growing interest in investigating non-linear evolution in spite of the reasons, such as those discussed in this paper and elsewhere [28], that have been brought forth against it. The feeling seems to be that an appropriate reformulation of the measurement process would eliminate the difficulties. Investigations by G. A. Goldin, H.-D. Doebner, and P. Nattermann [29, 30, 31] show that by a reasonable restriction on the set of allowed measurements, certain non-linear Schrödinger equations are then observationally equivalent, via a non-linear "gauge transformation" to the free linear equation. This shows that non-linearity *per se* may not lead to superluminal signals under a reasonably modified measurement postulate. As the equation studied by these authors are non-relativistic, we are still far from understanding the true relation of linearity to relativity.

3 Quantum indeterminism and superluminal communication.

The so called "quantum indeterminism" also seems to be connected with spatio-temporal relations. To see this, still under the Neighborhood Hypothesis, consider the already familiar EPR situation for the singlet two-photon states: there is a photon in each arm of the apparatus, and the combined state is the singlet: $(1/\sqrt{2})(H \otimes V + V \otimes H)$ where V indicates vertical linear polarization and H the orthogonal horizontal one. Consider a linear polarizer. The usual assumption is that the passage of a non-polarized photon (such as one of the photons of the singlet state) through this polarizer is absolutely random. If we write down a sequence of 0's and 1's where 0 means the photos is absorbed, and 1 passed through, then this sequence is thought to be at least a von Mises random sequence [32] in that no computable function that tries to predict the next outcome on the basis of outcomes already seen does better than chance. Suppose now that besides such a "randomizing" polarizer one also has a non-randomizing one for which some computable predictor does better than chance. Let one person then stay on arm A of the EPR apparatus armed with the two kinds of polarizers, which he may place in a horizontal orientation. Let his colleague stay at arm B with a polarizer placed in the vertical position, and let him use the computable predictor to guess whether the photon passes or not. Due to the strict mirror correlations in the singlet state, the sequence seen at arm B is the same one that is seen at arm A. Suppose that at arm A the randomizing polarizer is placed, then at arm B the predictions are seen to be no better than chance. Now one places the non-randomizing polarizer at arm A and shortly thereafter one sees at arm B better than chance predictions. If the arms are sufficiently distant this constitutes superluminal communication. Thus if quantum indeterminism exists in part of the world it must be universal if superluminal communication is to be ruled out. Once again, under the assumption of the impossibility of superluminal communication, distant quantum correlations have the effect of universalizing gross features of quantum mechanics. As before, it is not at all clear at this stage if the impossibility of superluminal communication, *per se*, imposes von Mises randomness on quantum events.

4 The second law of thermodynamics and superluminal communication

In Sections 2 and 3 we showed that assuming the impossibility of superluminal communication, and given the projection postulate, one can deduce that certain gross features of the quantum formalism must be universal and not admit any deviation. In this section we explore the relation of the hypothesis to thermodynamics by showing that given lorentz covariance, the second law of thermodynamics precludes non-entropic superluminal communication.

The second law and the hypothesis of the impossibility of superluminal communication have many superficial similarities. The most important of these is that both are statements about general physical objects of arbitrary size and complexity, stating that no such object can perform a certain task. In the case of the second law the task would be (among many equivalent formulations) to extract useful energy from heat, in the second case that of transmitting a message across a space-like interval. Superluminal communication *per se* is innocuous and only becomes problematic if its underlying physics is required to be lorentz covariant. In this case, it is well known that one can then send a message from a point on a time-like world-line to another point on the same world-line in the causal past of the first one (communication to one's past). It would then be possible for me to receive a message signed by me and dated tomorrow that reads: "Under no circumstances send this message!" Believing in free will, I take a firm resolve to obey the request. Am I then coerced in spite of my resolve to send it? If so, what happened to free will? If not, how did I happen to receive it? Superluminal communication in a lorentz covariant world is fraught with such causality paradoxes. In what follows we shall hypothetically assume such retrograde messaging in contexts where the paradoxical nature is either minimized or absent, tacitly supposing that its more bizarre manifestations get somehow resolved or avoided through mechanisms or circumstance yet unspecified. Presumably a consistent lorentz-covariant mathematical theory involving superluminal communication might resolve the paradoxes by imposing adequate boundary conditions on the solutions of the dynamic equations.

The second law is also about entropy, saying that it does not decrease in closed systems. Entropy is intimately tied to information. Communication is the transfer of information. It is then not surprising that if information can flow from the future to the past, reversing its normal direction, then

entropy in a closed system where such messaging takes place can be made to decrease, also reversing its normal direction.

In contemplating the thermodynamics of a system involving information transfer, it is necessary to take into account the entropy created in the process. Now a functioning communication channel should not in principle create entropy. To be clear, we are here considering the channel mechanism itself and not the processes involved in getting the message into and out of the channel. All such known channels involve the transfer of energy or matter (think of television or the postal service) and actually create entropy, but this entropy is incidental to the actual information transferred and comes about through inefficiencies that in principle can be eliminated or minimized below any preassigned level. One can communicate by a conservative and purely mechanical device. All that is needed is that the observed motion, invoked by the sender, have a meaning to the receiver. There are no known channels of superluminal communication. The hypothesized schemes or involve tachyons whose existence has not been established, or EPR-type correlations which cannot be used for communication unless, as was shown in Section 2, the hilbert-space quantum formalism breaks down, for which likewise there is no evidence. But even in such hypothesized channels there is no evident reason for energy degradation to be necessary. We shall therefore assume initially that our putative superluminal communication process does not create entropy, or at least that such entropy can, for a given message, be reduced as much as necessary. There *is* though some unavoidable entropy creation involved in *observing* a system (to get the necessary information that will become the content of the message) as has been pointed out by Brillouin [33, 34]. As the neglect of this type of entropy would lead to second-law violation, via a Maxwell's demon type argument, even without positing superluminal communication, these entropy sources must be considered.

We shall begin our discussion then with the assumption of a *non-entropic* superluminal communication process whose underlying physics is lorentz covariant. We then construct a communicator that sends messages to the past. This generally is achieved by signaling between different inertial frames, but this can be done by conventional means and so we can continue to posit that no entropy is created.

There are many (unworkable) proposals to foil the second law using thermodynamic fluctuations and one-way mechanical devices (ratchets, valves, etc.). The idea is that the device would be activated by a fortuitous but inevitable fluctuation, and the one-way nature of the device would prevent a re-

verse motion thereby storing energy (in a spring say) which is then extracted and used. One need only then wait for the next fluctuation to continue the process of extracting useful energy from heat. All such schemes fail. They *seem* plausible since one generally overlooks details of exactly what happens when the fluctuation activates the device. Surprisingly enough many, if not all, such proposals can be made to work if one can anticipate events, and retrograde communication makes this possible. Let us contemplate then one such proposal. Consider an enclosed rectangular volume with a rigid partition in the middle dividing the volume into two chambers (denoted by A and B) filled with a gas at the same pressure and temperature. Let us put a small one-way valve on the partition, the mobile element of which is held in place by a spring. If the local pressure on side A of the valve exceeds that on side B by some threshold difference then the force of the spring is overcome and the valve opens. Pressure excess on the other side has no effect due to the one-way nature of the mechanical construction. The (fallacious) argument is that due to pressure fluctuations the valve would open from time to time and allow gas molecules to pass from the supposedly (local) higher pressure side to the lower pressure side transferring molecules from chamber A to B and building up a true pressure (and temperature) difference violating the second law. The argument falls down because the molecules that push the valve open have already recoiled and are not the ones that actually pass through it. These, passing from both sides, have such a distribution of velocities that no pressure difference is built up. If however it were known beforehand *when* the valve would be pushed open, then one can simply temporarily remove the valve, exposing thus an orifice on the partition, and then the exchange of molecules across the orifice would be such that those passing from A to B would have a larger mean velocity than those passing in the other direction. Such information can be had if retrograde communication is possible and the second law can then be violated. This is essentially a variant on the Maxwell's demon scheme. The usual Maxwell's demon cannot violate the second law, since the process of observing the velocity of molecules, if done by actual photons, creates more entropy than the decrease achieved in operating the orifice closure to selectively let molecules pass from one side to the other [33]. The observing photon is necessarily of higher energy than the mean kinetic energy of the molecules and this thwarts the attempt. Our demon (observer) does not observe the molecules, all he need become aware of is a normal consequence of a fluctuation, such as a valve opening, and this can be done with low energy photons since the dimensions of the valve are

much larger than those of the molecules and it's motion is much slower.

This scenario does have its paradoxical aspects (inevitable consequence of the conjunction of lorentz covariance and superluminal communication). If the observer has free will and receives the correct information that at some instant the valve will be pushed open, then by removing the valve during a small time interval he destroys the chance of observing the valve being activated. He can still send the message to his past to maintain consistency with the fact of having received it (not doing so creates an even bigger paradox) but then the message has no real basis in observed events. This is self-consistent but hardly satisfying. To maintain some semblance of the ordinary order of things one can assume that the valve itself has an aperture that can be opened and closed and that it is small relative to the size of the valve so that even with the aperture open, pressure fluctuations still activate the valve. The observer now only send a message if he sees the valve activated (having already also opened the aperture by request of the message received). This still works as fluctuations have a certain size which gives a greater probability to transferring, via the aperture, higher velocity molecules from A to B and lower velocity ones in the opposite direction than to doing the opposite. Having reached this stage we can now actually eliminate the Brillouin "observer." One can automate the whole communication and valve operating process. The mechanism that receives the retrograde message can be coupled directly to the mechanism that opens the orifice. The message content itself can be reduced to a single bit, whose meaning would be "exactly T seconds from now open the orifice". One would need to keep an accurate clock and operate a device based on it, but this again in principle need not create entropy. The whole gadgetry surrounding the retrograde communicator can be taken to be a conservative system. The act of observing the valve can be eliminated by mechanically coupling the valve mechanism itself to the communication system. At this point the paradoxical nature of retrograde communication is practically absent. Our Rube-Goldberg-like gedanken experiment should of course be taken as a sort of visualizable enactment of a physical process that would be governed by a precise mathematical theory (using both advanced and retarded potentials, say), if indeed it were desirable to construct such a theory.

The assumption that the superluminal communication channel does not create entropy is crucial to the above argument. One may think that this hypothesis is not essential, arguing that what one need do is simply operate the scheme in those circumstances where the entropy decrease achieved (by

anticipating an adequate fluctuation, say) is greater than the entropy gain in using the channel. Although with specific assumptions about entropy created by the channel one can carry this through, there doesn't seem to be any *general* argument, at least we haven't been able to find one. We therefore tentatively conclude that one must necessarily impose restrictive assumptions on entropy creation in the channel in order to violate the second law.

Let us now summarize the hypotheses that we are relating: (LC), lorentz covariance; (SL), the second law of thermodynamics; (ISC), the impossibility of superluminal communication; (INESC), the impossibility of non-entropic superluminal communication, and (UQFCH), the universality of the quantum formalism with the collapse hypothesis.

We have finished showing: (LC and $\sim INESC \Rightarrow \sim SL$) which we prefer to state as: (LC and $SL \Rightarrow INESC$). In Section 2 we have proved: ($ISC \Rightarrow UQFCH$). In contrapositive the last two implications read: ($\sim UQFCH \Rightarrow \sim ISC$) and ($\sim INESC \Rightarrow \sim LC$ or $\sim SL$). What prevents linking these into a single implication is of course the non-entropic hypothesis, but in any case the two results clearly show what quantum breakdown entails.

Suppose that one makes an experimental test of quantum mechanics and finds a discrepancy between the experiment and the quantum prediction. This would typically be of the form $< R >^{\phi}_{exp} \neq (\phi, A_R\phi)$ where the left-hand side is the experimental mean value of some observable R in the state ϕ, and A_R is the self-adjoint operator that has been associated to this observable. Now one may be able to find a self-adjoint operator B_R such that for all the states tested one has $< R >^{\phi}_{exp} = (\phi, B_R\phi)$. In this case the discrepancy would most probably be explained by saying that, for some reason or other, the relation previously established between the observable R and A_R was in fact wrong and the correct relation should be to the operator B_R. Quantum mechanics would be preserved, "discrepancy" would then become "effect" and receive some usual explanation within the formalism as due to some new interaction energy, particle, etc. As such effects accumulate one can even foresee the possibility of a "keplerian" revision in which an equivalent formalism gives an account of the observed physics in a more elegant simplified form, but this would not be a true subversion of quantum mechanics as we know it. If however the observed phenomena *cannot* be explained by using a different self-adjoint operator, then, under the assumptions of Section 2, superluminal communication becomes reality. Once this happens, causal paradoxes arise if one is to maintain lorentz covariance. If in addition the superluminal channel is non-entropic, then one must definitely

abandon either lorentz covariance or the second law. In any case lorentz covariance is seriously threatened. Barring this, the projection postulate must be modified. In short, quantum mechanics will either persist essentially in its present form with absolutely no discernible deviation, or there will be a radical revision of physics. There cannot be any small revision. Quantum theory, thermodynamics, and lorentzian space-time are so interrelated that what affects one affects all.

And independent argument relating the second law of thermodynamics to locality (absence of superluminal influences) has been brought forward by Elitzur [35], without however expliciting the non-entropic assumption concerning the mechanisms of the hypothetical influence.

5 The covering law and superluminal communication.

The results of Section 2 are "local" in nature in that they shows that no deviations of certain types from ordinary hilbert space quantum mechanics, no matter how small, can be allowed if the impossibility of superluminal communication is to be maintained. But one is then immediately led to ask the interesting "global" question as to whether the impossibility of superluminal communication can be used as an axiom leading to hilbert space itself. This would provide a truly physical basis for hilbert space quantum theory motivated by the space-time condition of lorentz covariance. In this section we present some preliminary results in quantum logics that show that such a thesis is plausible. In a subsequent publication we shall present further evidence, which does not appeal to signals, connecting lorentz covariance and the hilbert space structure. We know from Piron [17] that a generalized hilbert space model can be constructed if the quantum logic obeys the so-called covering law. It is this law that we shall try to substantiate by space-time considerations. A possible relation between the covering law and the non existence of superluminal signals was also independently postulated by Nicolas Gisin (private correspondence) who derived the covering law from a condition similar to our CNS below.

Now whereas in Section 2 given the collapse hypothesis, we derive facts about detectors from the structure of hilbert space, and the hypothesis of the impossibility of superluminal communication, to prove anything in the

reverse direction seem to be impossible without positing something about the detection process, for otherwise we can simply and circularly *define* detections as those processes that do not allow long-range correlations to be used to send superluminal signals. This means that one must append to the axioms of quantum logic some postulates about the detection process. Fortunately, enough such expositions already exist in the literature and we shall simply appeal to some of these introduced by Guz [36, 37, 38] to make our point. In this we adopt Guz's notation to facilitate comparison with the cited works.

We start off conventionally with an orthomodular poset L of physical "propositions", and a convex set S of probability measures on L which are to represent "physical states". The set P of extreme points of S correspond to pure states. We assume that to each pure state p there is an atomic "indicator proposition" $s(p)$ that singles it out in that $p(s(p)) = 1$ and $q(s(p)) < 1$ for $q \neq p$. The map $p \mapsto s(p)$ is assumed to provides a bijection between the pure states and the atoms of L.

Measurements are to be described by a "collapse" scheme pretty much modeled on the notion of an ideal measurement in the conventional quantum formalism. In this view, if a proposition a is tested in a pure state p and found to be true, then the state is transformed ("collapses") into a new pure state p_a. This collapse occurs with frequency $p(a)$. This scheme is subject to the following conditions:

1. Propositions that test with certainty do not collapse the state. That is, $p_a = p$ whenever $p(a) = 1$. In particular, $p_1 = p$ and $p_{s(p)} = p$.

2. A repeated measurement yields the same results. That is, $s(p_a) \leq a$.

For two pure states p, q the number $(p : q) = p(s(q))$ is called the *transition probability* of p to q and corresponds to the fraction of times p collapses to q when tested by the indicator proposition of this last state. One sees that $(p : q) = 1$ implies that $p = q$.

We shall assume that given a set of pair-wise orthogonal propositions $\{b_1, \ldots, b_n\}$ there is a physical apparatus that tests them simultaneously with mutually exclusive outcomes. If the state is $p \in P$ then the outcome that renders proposition b_j true and all the others false occurs with frequency $p(b_j) = (p : p_{b_j})$. Note that in this case the "detection rate" of the whole apparatus is $\sum_{j=1}^{n}(p : p_{b_j}) = \sum_{j=1}^{n} p(b_j) = p(b_1 \vee \cdots \vee b_n)$. If $\bigvee_{j=1}^{n} b_j = 1$,

then after the state passes through the apparatus it becomes the *mixed* state

$$\sum_{j=1}^{n}(p : p_{b_j})p_{b_j}.$$

Let us now contemplate again an EPR-type space-time situation in which one has a state p, at one site A an apparatus corresponding to a proposition $a \in L$, and at a distant site B an apparatus corresponding to the pair-wise orthogonal set of propositions $\{b_1, \ldots, b_n\}$ with $\bigvee_{j=1}^{n} b_j = 1$. The arrangement is to operate in such a way that the events corresponding to registries in the apparatuses are space-like separated. We thus assume that the propositions a along with the b_j form a commuting set. For there to be no signals from site B to site A due to correlations present in state p (we will call this the "no-signal hypothesis"), the detection rate at A must be independent of the apparatus used at site B.

Now it doesn't seem possible to deduce the covering law just from the no-signal hypothesis. An extension of this hypothesis however to any situation formally as above where only commutativity (and not just space-like separation) is assumed does lead to the covering law.

> Commutative no-signal hypothesis (CNS): *Let $a, b_1, \ldots, b_n \in L$ be a commutative set and suppose the b_j pair-wise orthogonal with $\bigvee_{j=1}^{n} b_j = 1$, then $\sum_{j=1}^{n}(p : p_{b_j})p_{b_j}(a)$ is independent of the set $\{b_1, \ldots, b_n\}$.*

This can be viewed as saying there is no "statistical quantum contextualism", that is, the detection rate of a proposition is independent of what compatible mutually exclusive and exhaustive set of propositions one has measured just prior to it. If this set of propositions is measured at a space-like separated site, this follow from the no signal hypothesis.

Now one particular set of b_j that one can pick is the singleton $\{1\}$ for which the above sum is $(p : p_1)p_1(a) = p(a)$ and so the number whose independence is posited has to be $p(a)$. Let now a and b be two commuting propositions, then from the CNS we deduce:

$$(p : p_b)p_b(a) + (p : p_{b'})p_{b'}(a) = p(a)$$

$$(p : p_{b\wedge a'})p_{b\wedge a'}(a) + (p : p_{b\wedge a})p_{b\wedge a}(a) + (p : p_{b'})p_{b'}(a) = p(a)$$

from which using the fact that $p_{b\wedge a'}(a) = 0$ and $p_{b\wedge a}(a) = 1$ we deduce: $(p : p_b)p_b(a) = (p : p_{b\wedge a})$ which is the same as

$$(p : p_b)(p_b : (p_b)_a) = (p : p_{a\wedge b})$$

which says that the detection rate with the successive observations of b followed by a is equal to the detection rate of the observation of $a \wedge b$. By complete symmetry we also have $(p : p_a)(p_a : (p_a)_b) = (p : p_{a\wedge b})$.

Consider now $b = s(q_a)$ where q is any other state. As $b \leq a$, the two commute and we can apply the above results. Now, since b is an atom, for any state r one has $r_b = s^{-1}(b) = q_a$ provided $r(b) \neq 0$. Assume $p(b) \neq 0$, then by the last formula of the previous paragraph one deduces $(p : p_a)(p_a : q_a) = (p : q_a)$. If now $(p : p_a) = (p : q_a)$ one concludes that $(p_a : q_a) = 1$ which implies that $p_a = q_a$ thus one has shown

$$(p : p_a) = (p : q_a) \neq 0 \Rightarrow p_a = q_a.$$

What is interesting about this result is that this is precisely the property one needs in the axiomatic scheme presented by Guz [38] to deduce the covering law for the poset L. Thus if one can somehow justify the extension of necessary conditions on space-like separated measurement to generally commuting measurements, the covering law would follow from the no-signal hypothesis. Lacking this, we have a weaker result in that a stronger hypothesis (CNS), leads to the covering law in at least one existing axiomatic framework. This axiomatics was not, to our awareness, conceived to establish a connection between space-time structure and the covering law and as such may not be the best to make such a connection plausible. We shall address this problem in a future publication. There is a general weakness in all present axiomatic schemes for quantum mechanics, they do not distinguish space-like separated propositions from other commutative pairs, for the only relation that one normally posits between space-time and quantum logic is that space-like separation leads to commutativity. Only by a joint axiomatization of both lorentzian space-time and hilbert space quantum mechanics can one hope for anything better. Now commutativity is generally interpreted as "commensurability" and this means the ability to make simultaneous measurements which in turn suggests that space-like separations always play some role in commutativity, yet this has never been thoroughly examined. Furthermore if one believes in general unitary symmetry in hilbert-space, it's not hard to find examples in which a commutative

pair of observables pertaining to a single localized system (a bound pair of particles say) is unitarily equivalent to a pair of observables at the opposite arms of an EPR apparatus. Such a symmetry would thus extend the requirements on space-like commutativity to commutativity in general. That arguments pertaining to space-like commutativity can be carried over to general commutativity has already been pointed out by Home and Sengupta [39] in relation to Bell's inequalities. It may thus well be that the weaker result with the CNS assumption is not too distant from the desired strong result especially if one imposes strong symmetry requirements. In any case, it is clear that one cannot hope to reach a full understanding of hilbert-space quantum mechanics without linking it to space-time structure.

6 Conclusions

In previous sections we gave some arguments in support of the thesis that, given certain hypotheses, standard quantum mechanics is structurally unstable, and neighboring theories generically run into difficulties with relativity. This spells trouble for those that feel that standard quantum mechanics is only an approximation and propose alternatives. Great care must be taken in formulating these alternatives. Some have even come to feel that all alternatives are ruled out. Stephen Weinberg in his book [40] has tentatively reached this conclusion:

> ...I could not find any way to extend the nonlinear version of quantum mechanics to theories based on Einstein's special theory of relativity ... both N. Gisin in Geneva and my colleague Joseph Polchinski at the University of Texas independently pointed out that ... the nonlinearities of the generalized theory *could* be used to send signals instantaneously over large distances... At least for the present I have given up on the problem; I simply do not know how to change quantum mechanics by a small amount without wrecking it altogether.
>
> This theoretical failure to find a plausible alternative to quantum mechanics, ... suggest to me that quantum mechanics is the way it is because any small change in quantum mechanics would lead to logical absurdities. If this is true, quantum mechanics may be a permanent part of physics. Indeed, quantum mechanics may

survive not merely as an approximation to a deeper truth, ... but as a precisely valid feature of the final theory.

A careful analysis of what really goes into the perception of this rigidity reveals the striking role played by the projection hypothesis or some modification thereof. This hypothesis in turn is based on the notion of instantaneous physical state which obviously is a notion tied to a given inertial frame. It is this frame-dependence that cannot be reconciled with relativity in the alternative theories. Abandoning such a frame-dependent notion would mitigate arguments against alternatives and opens up a true possibility for changing quantum mechanics "by a small amount" (as far as numerical predictions are concerned) without "wrecking it altogether" (maintaining relativity). Weinberg's reasons for his final speculation on the survival of quantum mechanics may in the end not be all that compelling.

In recent years there has emerged a new quantum mechanical formalism (along with several interpretations of it) based on consistent histories and decoherence [41, 42], that essentially does away with the projection postulate and to a large extent with the notion of instantaneous state. As such, it escapes the analysis of this paper and shows a very promising possibility for those that wish to modify quantum mechanics and still maintain special relativity. We shall investigate these questions in a future publication.

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A Appendix: The theorem

We start with some notation and preliminaries. Our hilbert spaces are complex and finite dimensional. For a hilbert space H we denote by $L(H)$ the space of operators in H, and by I the identity map. If $e \in H$ we denote by $|e|$ the equivalence class of e under the relation by which two elements e and f are equivalent if and only if $e = \theta f$ for some unimodular complex number θ. We denote by S_H the resulting quotient space. A positive rank-one operator is necessarily of the form $(e, \cdot)e$ with e determined only up to a

unimodular multiple. Thus the space of positive rank-one operators is in a bijective correspondence with S_H minus the class of the zero vector. We shall reserve the Greek letter ρ to represent a positive rank-one operator. A map $G : S_H \to S_K$ can be lifted in infinitely many ways to a map $\hat{G} : H \to K$ by associating to $e \in H$ an arbitrarily chosen element of the equivalence class $G|e|$. We say that $\hat{G}$ *represents* G. Two such representations $\hat{G}_1$ and $\hat{G}_2$ are related by $\hat{G}_1 f = \theta(f)\hat{G}_2 f$ for some unimodular function θ. If $\hat{G}$ can be chosen to be linear or antilinear, we say that G is *linearly* or *antilinearly representable*. We reserve the Greek letter θ to indicate unimodular functions. We denote by $[f_1, f_2, \dots, f_q]$ the linear subspace generated by the indicated vectors. We use the superscript star symbol $(\cdot)^\star$ to indicate hermitian conjugate and, when the bar is inconvenient, the complex conjugate of numbers.

Theorem 1 *Let H and K be complex finite-dimensional hilbert spaces and $W : L(H) \to L(K)$ a linear map. Suppose W maps a positive rank-one operators to either a positive rank-one operators or zero, then its action on positive rank-one operators is either of the form $W\rho = C\rho C^\star$ where $C : H \to K$ is linear or antilinear, or $W\rho = Tr(D\rho)(k, \cdot)k$ for some positive operator $D \in L(H)$ and $k \in K$.*

We shall call the three kinds of transforms as being of the *linear, antilinear* and *degenerate* type.

This theorem is a relative of Wigner's theorem [43] which we use in part of the proof.

Proof: If $k_1, \dots, k_n$ is an orthonormal basis for K, then one can write the action of W as

$$WA = \sum Tr(AL_{ij})(k_i, \cdot)k_j$$

for some $L_{ij} \in L(H)$.

The action of W on rank-one positive operators can be considered as a map $S_H \to S_K$. Let $T : H \to K$ be a map, that represents this action. One easily shows that replacing L_{ij} with $\frac{1}{2}(L_{ij} + L_{ji}^\star)$ one has the same action on S_H and so we can assume that $L_{ij}^\star = L_{ji}$, in particular the diagonal elements L_{ii} are self adjoint. We have $|(k_i, Tf)|^2 = (f, L_{ii}f)$, so each L_{ii} is a positive operator and if we let $L = \sum L_{ii}$, we have $\|Tf\|^2 = (f, Lf)$. By the spectral theorem, there is a positive invertible operator M such that MLM is an orthogonal projector. By working with TM instead of T, one can assume that L is a projector and we do so.

Now if $\dim H = 0$ there is nothing to investigate. If $\dim H = 1$, generated by the unit vector e, then the L_{ij} are plain numbers. By performing a unitary change of basis in K, the matrix (L_{ij}) undergoes a unitary similarity transformation and since it is self-adjoint, can be brought into diagonal form. Since its rank must be at most one, it is either zero and there is nothing more to consider, or it has a diagonal form with $L_{11} = 1$ and the rest of the elements zero. In this last case we have $W\rho = C\rho C^\star$ where $C : H \to K$ is the linear map defined by $Ce = k_1$ and the theorem is true in this case.

Thus we next investigate the case that $\dim H = 2$. If $\dim K < 2$ we can extend K by a direct summand to achieve dimension 2, and extend T by zero into this summand. The case of $\dim K > 2$ we leave for later. For $\dim K = 2$ we choose orthonormal bases in H and K and identify both spaces with $\mathbb{C}^2$.

One now has:

$$(L_{ij}) = \begin{pmatrix} \alpha I + \vec{\beta} \cdot \vec{\sigma} & AI + \vec{B} \cdot \vec{\sigma} \\ A^\star I + \vec{B}^\star \cdot \vec{\sigma} & \gamma I + \vec{\delta} \cdot \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the usual Pauli spin matrices. Now L_{11} and L_{22} sum to a projector. If this is I then $L_{22} = I - L_{11}$, if of rank one, then,being positive, both L_{11} and L_{22} must be proportional to it, and if zero, both must be zero. In any case there is a basis of H in which both are diagonal and so we can assume that both $\vec{\beta}$ and $\vec{\delta}$ have only the third component non zero which we shall at an appropriate point denote simply by β and δ. Let u be a vector in $\mathbb{C}^2$ of norm one, then $u^\star \vec{\sigma} u$ is a unit vector in $\mathbb{R}^3$ and any such vector can be so constructed. Denoting such a vector by $\hat{n}$ one has that $(Tu, \cdot)Tu$, in the chosen basis, is given by the following matrix:

$$\begin{pmatrix} \alpha + \vec{\beta} \cdot \hat{n} & A + \vec{B} \cdot \hat{n} \\ A^\star + \vec{B}^\star \cdot \hat{n} & \gamma + \vec{\delta} \cdot \hat{n} \end{pmatrix}$$

This matrix must have rank at most one for any choice of $\hat{n}$ and so it's determinant must vanish. This gives

$$(\alpha + \vec{\beta} \cdot \hat{n})(\gamma + \vec{\delta} \cdot \hat{n}) = (A + \vec{B} \cdot \hat{n})(A^\star + \vec{B}^\star \cdot \hat{n})$$

for all $\hat{n}$. In deriving conclusions from this expression one must be careful to symmetrize the coefficients of the quadratic terms in $\hat{n}$ and to subtract from them the trace contribution since one has $\hat{n} \cdot \hat{n} = 1$. This now gives us the following relations:

$$\alpha\gamma + \frac{1}{3}\vec{\beta} \cdot \vec{\delta} \;\; = \;\; |A|^2 + \frac{1}{3}\vec{B}^\star \cdot \vec{B},$$

$$\begin{aligned} \alpha\vec{\delta} + \gamma\vec{\beta} &= A^\star \vec{B} + A\vec{B}^\star, \\ \beta_i\delta_j + \delta_i\beta_j - \frac{2}{3}\vec{\beta}\cdot\vec{\delta}\delta_{ij} &= B_i^\star B_j + B_i B_j^\star - \frac{2}{3}\vec{B}^\star\cdot\vec{B}\delta_{ij}. \end{aligned}$$

If now L is zero then both L_{11} and L_{22} are zero and we deduce from the first equation that $L_{ij} = 0$ and there is nothing more to prove. We now use the fact that $\beta_i = \beta\delta_{i3}$ and $\delta_i = \delta\delta_{i3}$ The above equations now give rise to:

$$\begin{aligned} \alpha\gamma + \frac{1}{3}\beta\delta &= |A|^2 + \frac{1}{3}\vec{B}^\star\cdot\vec{B}, && (1) \\ \alpha\delta + \gamma\beta &= 2\,\mathrm{Re}\,(A^\star B_3), && (2) \\ \mathrm{Re}\,(A^\star B_i) &= 0, \qquad (i \neq 3), && (3) \\ 2\beta\delta &= 2|B_3|^2 - |B_1|^2 - |B_2|^2, && (4) \\ \mathrm{Re}\,(B_i^\star B_3) &= 0, \qquad (i \neq 3) && (5) \\ -\beta\delta &= 2|B_1|^2 - |B_2|^2 - |B_3|^2, \\ &= 2|B_2|^2 - |B_1|^2 - |B_3|^2, && (6) \\ \mathrm{Re}\,(B_1^\star B_2) &= 0. && (7) \end{aligned}$$

From (6) one deduces that $|B_1| = |B_2|$. Now one still has the freedom to make a unitary change of basis in H by a diagonal matrix with unimodular entries. The effect of this is to perform a *real* rotation of the (B_1, B_2) vector. In this way one can choose the imaginary part of B_1 to be rotated to zero. Thus we assume $B_1 = B$ a real number, and by (7) one has $B_2 = \pm iB$. There are now two cases:

Case I: $B \neq 0$

From (3) we have $A = 0$ and from (5) $B_3 = 0$. Equations (4) and (6) are now equivalent and give $\beta\delta = -B^2$ and this in equation (1) gives $\alpha\gamma = B^2$. Using these in equation (2) we deduce that $\alpha^2 = \beta^2$ and $\gamma^2 = \delta^2$. Now since the L_{ii} are positive operators we have that $\alpha \geq 0$ and $\gamma \geq 0$. The projector L has the matrix:

$$\begin{pmatrix} \alpha+\gamma+\beta+\delta & 0 \\ 0 & \alpha+\gamma-\beta-\delta \end{pmatrix}.$$

Since $B \neq 0$ neither α nor γ is zero. Using now the fact that α and γ are positive, that β and δ have opposite signs, and that $\alpha^2 = \beta^2$ and $\gamma^2 = \delta^2$ we deduce that neither diagonal entry in the matrix can be zero. Thus L must be the identity. This means that $\beta = -\delta$ and so all four numbers must have the same modulus. Exchanging the two basis elements in K exchanges the role of

L_{11} and L_{22}, thus without loss of generality we can put $\alpha = \beta = \gamma = -\delta = \frac{1}{2}$. Changing the sign of one of the basis elements of H results in changing the sign of the (B_1, B_2) vector so we can choose B to be positive, in which case it must be $\frac{1}{2}$ by previous equations. Thus this case reduces to the following canonical forms:

$$\frac{1}{2}\begin{pmatrix} I+\sigma_3 & \sigma_1 \pm i\sigma_2 \\ \sigma_1 \mp i\sigma_2 & I-\sigma_3 \end{pmatrix}$$

Case II: $B = 0$

A unitary change of basis in K causes the representation for (L_{ij}) to change by subjecting the two matrices

$$\begin{pmatrix} \alpha & A \\ A^\star & \gamma \end{pmatrix} \qquad \begin{pmatrix} \beta & B_3 \\ B_3^\star & \delta \end{pmatrix}$$

to the same unitary similarity transformation. The two matrices are hermitian, and so there is no loss in generality in assuming that the second is diagonal and thus $B_3 = 0$. The two now equivalent equations (4) and (6) give $\beta\delta = |B_3|^2 = 0$, and then from (1) we get $\alpha\gamma = |A|^2$. There are now two subcases:

Subcase IIa: L is rank one.

Without loss of generality assume that the second entry of L is zero, then we have $\alpha + \gamma = \beta + \delta$ and since the first entry must be one, one has that both sums must be $\frac{1}{2}$. Since $\beta\delta = 0$ one of the two must be zero and the other $\frac{1}{2}$. Without loss of generality we set $\delta = 0$. Equation (2) now gives $\gamma = 0$ which finally implies that $A = 0$ and we have the canonical form:

$$\frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & 0 \end{pmatrix}$$

Subcase IIb: $L = I$

We now must have $\beta + \delta = 0$ which with $\beta\delta = 0$ means that both are zero. Thus the second one of the displayed pair of matrices above is zero and we are free to diagonalize the first one. In this case, from $\alpha\gamma = |A|^2$ one sees that either α or γ must be zero and the other 1. Without loss of generality, assume $\alpha = 1$, and we have the final canonical form:

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

Let us now examine the cases. One of the forms of Case I corresponds to $W\rho = I\rho I$ and the other to $W\rho = J\rho J$ where J is the antilinear involution

on $\mathbb{C}^2$ given by $J(u_1, u_2) = (\bar{u}_1, \bar{u}_2)$. The form of Case IIa corresponds to $W\rho = P\rho P$ where P is the projector on the first component. The last form cannot be expressed via a linear or antilinear transformation and corresponds to $W\rho = Tr(\rho)(k_1, \cdot)k_1$. We see here the three types mentioned at the beginning: the *linear, antilinear* and *degenerate*.

If we take into account the changes of basis that were made, and the fact that L was changed into a projector by considering TM instead of T we see that the general (non-canonical) form for the degenerate type (still with both dimensions two) is $Tf = \theta(f)(f, Df)^{1/2}k$ for some positive definite operator D and some $k \in K$.

Consider now the general case of arbitrary dimension greater or equal to two for each hilbert space. Choose a two-dimensional orthogonal projector Q in K and a two-dimensional orthogonal projector P in H. One has that QTP is one of the three types. These local representations will now be joined into a global one.

Firstly we shall keep P fixed, and by joining the representations of the QTP, find a representation for TP. For simplicity's sake we shall generally omit writing P in the next two paragraphs.

Consider the degenerate type. Suppose that $K_0 \subseteq K$ is a subspace with the property that if Q_0 is the orthogonal projector on K_0, then $Q_0Tf = \theta(f)(f, Df)^{1/2}g$ for some $g \in K_0$ and some positive definite operator D. If $K_0 \neq K$, choose a vector h orthogonal to K_0. Let Q be the orthogonal projection onto the subspace $[g, h]$ and Q_g the projector onto $[g]$. Suppose that QT is either of the linear or antilinear type. Then one has from $Q_gQ_0Tf = Q_gQTf$ that $\theta(f)(f, Df)^{1/2} = \theta'(f)\phi(f)$ for some linear or antilinear form f. This is obviously impossible due to the positive definiteness of D. Thus the type must be degenerate and we have $QTf = \theta'(f)(f, D'f)^{1/2}k$ for some vector k and some positive definite operator D'. By the same token as before we must now have $\theta(f)(f, Df)^{1/2}g = \theta'(f)(f, D'f)^{1/2}Q_gk$. This means that $g = aQ_gk$ for some non-zero number a and $\theta'(f)(f, D'f)^{1/2} = a\theta(f)(f, Df)^{1/2}$ Let K_1 be the span of K_0 and h, and Q_1 the orthogonal projector onto it. From $Q_1 = Q_0 + Q - Q_g$, we get that $Q_1Tf = \theta(f)(f, Df)^{1/2}g + \theta'(f)(f, D'f)^{1/2}(I - Q_g)k = \theta(f)(f, Df)^{1/2}ak$ which is precisely the form that was assumed for K_0. By induction therefore, if any of the two-dimensional forms QT is degenerate then $Tf = \theta(f)(f, Df)^{1/2}k$, for some vector $k \in K$ and some positive definite operator D.

Suppose now that all the QT types are linear or antilinear. If the dimension of the linear span of the range of T in K is two or less then we

are through since for some two-dimensional projector Q we would have $TP = QTP$. Suppose therefore that there are at least three linearly independent vectors in the range and so choose three such k_1, k_2, k_3 that are pairwise orthogonal. Let Q_{ij} be the orthogonal projection on $[k_i, k_j]$. One has $Q_{ij}Tf = \theta_{ij}(f)(\phi_{ij}(f)k_i + \psi_{ij}(f)k_j)$ for linear or antilinear functionals, as the case may be, ϕ_{ij} and ψ_{ij}. In particular one has that ϕ_{12} and ψ_{12} are linearly independent and so ψ_{13} is a linear combination of the two or their complex conjugates, say $\psi_{13}(f) = a\phi_{12}(f) + b\psi_{12}(f)$ (the case $\psi_{13}(f) = a\overline{\phi_{12}(f)} + b\overline{\psi_{12}(f)}$ is analogous). Now when $\phi_{12}(f) = 0$, Tf is proportional to k_2 and so $(k_3, Tf) = 0$ which implies that $\psi_{13}(f) = 0$, and hence $b = 0$. One must thus have $\psi_{13}(f)$ be proportional to $\phi_{12}(f)$ or its complex conjugate. A similar reasoning leads to the conclusion that $\psi_{23}(f)$ is proportional to $\psi_{12}(f)$ or its complex conjugate. But $(k_3, Tf) = \theta_{13}(f)\psi_{13}(f) = \theta_{23}(f)\psi_{23}(f)$. This proportionality contradicts, by the linear independence of ϕ_{12} and ψ_{12}, the previously proved proportionalities unless both $\psi_{i3}, i = 1, 2$ are zero. This now contradicts the existence of the k_i. Thus the dimension of the span cannot be greater than two and this case is done.

We now pass to joining the local representations of TP to a global one of T.

Suppose that for a two-dimensional projector P one has that $TPf = \theta(f)(f, Df)^{1/2}g$ for some positive definite operator D and some vector g. Suppose that the range of T is not wholly contained in a one-dimensional subspace. Then there is a vector h with Th and g linearly independent. Consider the two dimensional subspaces $[f, h]$ where $f \in PH$. Since on these subspaces the image of T contains at least two linearly independent vectors, the type cannot be degenerate and we must have by either linearity or antilinearity that $T(f+\alpha h) = \theta_1(f, \alpha)Tf + \theta_2(f, \alpha)\alpha Th = \theta_1(f, \alpha)\theta(f)(f, Df)^{1/2}g + \theta_2(f, \alpha)\alpha Th$. (Note that a complex number and its complex conjugate differ by a unimodular factor, so both the linear and antilinear cases are subsumed in the last expression.) This now defines the action of T on the three dimensional subspace $PH \oplus [h]$. The coefficient of $(g, \cdot)Th$ in $(T(f+\alpha h), \cdot)T(f+\alpha h)$ must, by the general expression for W, be of the form $(f + \alpha h, N(f + \alpha h))$ for some linear operator N, and so we must have

$$\overline{\theta_1(f, \alpha)\theta(f)}\theta_2(f, \alpha)(f, Df)^{1/2}\alpha =$$
$$(f, Nf) + \alpha(f, Nh) + \bar{\alpha}(h, Nf) + |\alpha|^2(h, Nh)$$

Now the first and fourth term on the right-hand side must vanish since the left-hand side vanishes when either f or α vanishes. The modulus of the

left-hand side does not change if we multiply α by a unimodular number, and for this to be true of the right-hand side then, for a given f, one of the two middle terms must vanish. Thus as a functional of f, if one of (f, Nh), (h, Nf) is not the zero functional, then the other must vanish whenever the first one doesn't, which is only possible for the zero functional. Thus only one of the middle terms is not zero, which contradicts the positive definiteness of D since a linear or anti-linear functional always has a non-trivial kernel. Thus in our case the image of T is contained wholly in a one-dimensional subspace $[k]$ and one must have, by the general expression for W, taking $k_1 = k$, that $Tf = \theta(f)(f, L_{11}f)^{1/2}k$ on all of H.

We are now left in the last case of all the types of TP being linear or antilinear. Remember that we can assumed that L is a projector. For $f \in H$, one has $f = Lf + (I - L)f$. Assume both terms are not zero. On the two-dimensional subspace spanned by the two terms, T is either of the linear or antilinear type, thus $Tf = \theta_1(f)TLf + \theta_2(f)T(I - L)f$, but $\|T(I - L)f\| = \|L(I - L)f\| = 0$ Thus $|Tf| = |TLf|$ and so TL represents the same transformation $S_H \to S_K$ that T does. One thus need only consider T on LH and so assume that L is the identity (of dimension at least two, as the one-dimensional case was already settled). With this assumption, one has that on any two-dimensional subspace, T is representable by either a linear or antilinear isometry, and hence, by the polarization identity, preserves the moduli of inner products. Let $e_1, \ldots, e_n$ be an orthonormal basis for H, then the $Te_1, \ldots, Te_n$ are orthonormal. By linearity or antilinearity on two-dimensional subspaces we have $T(\sum_i^n \alpha_i e_i) = T(\alpha_1 e_1 + \sum_2^n \alpha_i e_i) = \theta(\alpha)\alpha_1 T(e_1) + \theta'(\alpha)T(\sum_2^n \alpha_i e_i)$. Continuing in this vein one has $T(\sum_i^n \alpha_i e_i) = \sum_i^n \theta_i(\alpha)\alpha_i T(e_i)$. Thus the range of T is totally contained in the subspace generated by the $Te_1, \ldots, Te_n$. By working only with the range, we can thus assume K and H are of the same dimension and so we identify them. By Wigner's theorem, any mapping of a hilbert space into itself which preserves the moduli of inner products is representable by either a unitary or an antiunitary operator. This completes the last case and the proof of the theorem. Q.E.D

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